

# Nordhaus-Gaddum bounds for locating domination

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## Abstract

A dominating set  $S$  of graph  $G$  is called *metric-locating-dominating* if it is also locating, that is, if every vertex  $v$  is uniquely determined by its vector of distances to the vertices in  $S$ . If moreover, every vertex  $v$  not in  $S$  is also uniquely determined by the set of neighbors of  $v$  belonging to  $S$ , then it is said to be *locating-dominating*. Locating, metric-locating-dominating and locating-dominating sets of minimum cardinality are called  $\beta$ -codes,  $\eta$ -codes and  $\lambda$ -codes, respectively. A Nordhaus-Gaddum bound is a tight lower or upper bound on the sum or product of a parameter of a graph  $G$  and its complement  $\overline{G}$ . In this paper, we present some Nordhaus-Gaddum bounds for the location number  $\beta$ , the metric-location-domination number  $\eta$  and the location-domination number  $\lambda$ . Moreover, in each case, the graph family attaining the corresponding bound is fully characterized.

**Keywords:** Domination, Location, Locating domination, Nordhaus-Gaddum

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## 1. Introduction

Given a graph  $G = (V, E)$ , the (*open*) *neighborhood* of a vertex  $v \in V$  is  $N_G(v) = N(v) = \{u \in V : uv \in E\}$ . The distance between vertices  $v, w \in V$  is denoted by  $d_G(v, w)$ , or  $d(v, w)$  if the graph  $G$  is clear from the context. The diameter  $diam(G)$  is the maximum distance between any two vertices of  $G$ . Let  $S = \{x_1, \dots, x_k\}$  be a set of vertices and let  $v \in V \setminus S$ . The ordered  $k$ -tuple  $c_S(v) = (d(v, x_1), \dots, d(v, x_k))$  is called the vector of *metric coordinates* of  $v$  with respect to  $S$ . For further notation see [4].

A set  $D \subseteq V$  is a *dominating set* if for every vertex  $v \in V \setminus D$ ,  $N(v) \cap D \neq \emptyset$ . The *domination number*  $\gamma(G)$  is the minimum cardinality of a dominating set of  $G$ . A dominating set of cardinality  $\gamma(G)$  is called a  $\gamma$ -code [8]. A set  $D = \{x_1, \dots, x_k\} \subseteq V$  is a *locating set* if for every pair of distinct vertices  $u, v \in V$ ,  $c_D(u) \neq c_D(v)$ . The *location number* (also called the *metric dimension*)  $\beta(G)$  is the minimum cardinality of a locating set of  $G$  [7, 14]. A locating set of cardinality  $\beta(G)$  is called a  $\beta$ -code. A *metric-locating-dominating set*, a *MLD-set* for short, is any set of vertices that is both a dominating set and a locating set. The *metric-location-domination number*  $\eta(G)$  is the minimum cardinality of a metric-locating-dominating set of  $G$ . A metric-locating-dominating set of cardinality  $\eta(G)$  is called a  $\eta$ -code [10]. A set  $D \subseteq V$  is a *locating-dominating set*, an *LD-set* for short, if for every two vertices  $u, v \in V(G) \setminus D$ ,  $\emptyset \neq N(u) \cap D \neq N(v) \cap D \neq \emptyset$ . The *location-domination number*  $\lambda(G)$  is the minimum cardinality of a locating-dominating set. A locating-dominating set of cardinality  $\lambda(G)$  is called a  $\lambda$ -code [15]. A complete and regularly updated list of papers on locating dominating codes is to be found in [13].

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Clearly, every locating-dominating set is locating and also dominating. Moreover, both location and domination are hereditary properties. Particularly, if for two sets  $S_1, S_2 \subset V$ ,  $S_1$  is locating and  $S_2$  is dominating, then  $S_1 \cup S_2$  is both locating and dominating. Hence, for every graph  $G$ ,  $\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \min\{\gamma(G) + \beta(G), \lambda(G)\}$  [2].

A Nordhaus-Gaddum bound is a tight lower or upper bound on the sum or product of a parameter of a graph  $G$  and its complement  $\bar{G}$  [1, 9, 12]. For example, in [5] it was shown that for any graph  $G$  of order  $n$ ,  $\gamma(G) + \gamma(\bar{G}) \leq n + 1$ , the equality being true only if  $\{G, \bar{G}\} = \{K_n, \bar{K}_n\}$ . In this paper, we present some Nordhaus-Gaddum bounds on the sum of the location number  $\beta$ , the metric-location-domination number  $\eta$  and the location-domination number  $\lambda$ . In all cases, the classes of graphs attaining both bounds are characterized.

## 2. Nordhaus-Gaddum bounds

Unless otherwise stated, along this section  $G = (V; E)$  is a, not necessarily connected, nontrivial graph of order  $n$ . A graph  $G$  is called *doubly-connected* if both  $G$  and its complement  $\bar{G}$  are connected. As usual,  $K_n$ ,  $C_n$  and  $P_n$  denote respectively the complete graph, the cycle and the path on  $n$  vertices.

### 2.1. Location number

**Theorem 1.** *For every nontrivial graph  $G$ ,  $2 \leq \beta(G) + \beta(\bar{G}) \leq 2n - 1$ . Moreover,*

- $\beta(G) + \beta(\bar{G}) = 2$  if and only if  $G = P_4$ .
- $\beta(G) + \beta(\bar{G}) = 2n - 1$  if and only if  $\{G, \bar{G}\} = \{K_n, \bar{K}_n\}$ .

*Proof.* Every graph satisfies  $1 \leq \beta(G)$ , which means that  $2 \leq \beta(G) + \beta(\bar{G})$ . Moreover, the equality  $\beta(G) + \beta(\bar{G}) = 2$  is only true for  $G = P_4$ , since paths  $P_n$  are the only graphs with location number 1 [3], and  $P_4 = \bar{P}_4$  is the only nontrivial path whose complement is also a path. The upper bound immediately follows from these facts: (1) the graph  $\bar{K}_n$  is the only graph with location number  $n$  and (2)  $\beta(K_n) = n - 1$ . Finally, claims (1) and (2) also allows us to derive that equality  $\beta(G) + \beta(\bar{G}) = 2n - 1$  only holds when  $\{G, \bar{G}\} = \{K_n, \bar{K}_n\}$ .  $\square$

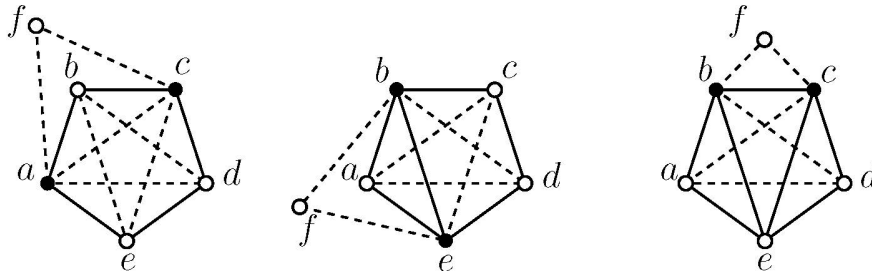


Figure 1: Solid lines are edges in  $G$  and dashed lines are edges in  $\bar{G}$ .

**Lemma 1.** Every doubly-connected graph  $G$  of order  $n \geq 6$  such that  $\text{diam}(G) = \text{diam}(\overline{G}) = 2$  contains a locating set of cardinality  $n - 4$ .

*Proof.* Let  $\rho$  be an induced path of order 4 in  $G$ , whose existence is guaranteed since, as was proved in [6], the complement of every nontrivial  $P_4$ -free graph is not connected. Assume that  $V(\rho) = \{a, b, c, d\}$  and  $E(\rho) = \{ab, bc, cd\}$ . Since,  $\text{diam}(G) = 2$ , there exists a vertex  $e \in V(G)$  such that  $d_G(a, e) = d_G(e, d) = 1$ . We distinguish three cases.

**Case 1:**  $eb, ec \notin E(G)$  (see Figure1, left). In this case, the set  $\{a, b, c, d, e\}$  determines an induced cycle  $\Gamma$  in  $G$  and also an induced cycle  $\overline{\Gamma}$  in  $\overline{G}$ . Let  $f$  a vertex not belonging to  $\{a, b, c, d, e\}$ . Either in  $G$  or in  $\overline{G}$ ,  $f$  has at most two neighbors in  $\{a, b, c, d, e\}$ . Without loss of generality we may suppose that  $N_G(f) \cap \{a, b, c, d, e\} \leq 2$  (otherwise we interchange labels  $G$  and  $\overline{G}$ ), which means that there exist in  $\{a, b, c, d, e\}$  a pair of non-consecutive vertices non-adjacent to  $f$ . Again w.l.o.g. we assume that  $N_G(f) \cap \{a, c\} = \emptyset$ . Certainly, the set  $V(G) \setminus \{b, d, e, f\}$  is a locating set of  $G$  since  $c_{\{a, c\}}(b) = (1, 1)$ ,  $c_{\{a, c\}}(d) = (2, 1)$ ,  $c_{\{a, c\}}(e) = (1, 2)$  and  $c_{\{a, c\}}(f) = (2, 2)$ .

**Case 2:**  $e$  is adjacent to exactly one vertex of  $\{b, c\}$ . Let us assume that  $eb \in E(G)$  and  $ec \notin E(G)$  (see Figure1, center). In this case,  $d_G(e, b) = 1$ , which means that  $d_{\overline{G}}(e, b) = 2$  since  $\text{diam}(\overline{G}) = 2$ . Therefore, there exists a vertex  $f \notin \{a, b, c, d, e\}$  such that  $d_{\overline{G}}(e, f) = d_{\overline{G}}(f, b) = 1$ . This means that  $d_G(e, f) = d_G(f, b) = 2$  as  $\text{diam}(\overline{G}) = 2$ . Hence, the set  $V(G) \setminus \{a, c, d, f\}$  is a locating set of  $G$  since  $c_{\{b, e\}}(a) = (1, 1)$ ,  $c_{\{b, e\}}(c) = (1, 2)$ ,  $c_{\{b, e\}}(d) = (2, 1)$  and  $c_{\{b, e\}}(f) = (2, 2)$ .

**Case 3:**  $eb, ec \in E(G)$  (see Figure1, right). Since  $d_G(b, c) = 1$ , we have  $d_{\overline{G}}(b, c) = 2$ . Therefore, there exists a vertex  $f \notin \{a, b, c, d, e\}$  such that  $d_{\overline{G}}(b, f) = d_{\overline{G}}(f, c) = 1$ . This means that  $d_G(b, f) = d_G(f, c) = 2$ . Hence, the set  $V(G) \setminus \{a, d, e, f\}$  is a locating set of  $G$  since  $c_{\{b, c\}}(a) = (1, 2)$ ,  $c_{\{b, c\}}(d) = (2, 1)$ ,  $c_{\{b, c\}}(e) = (1, 1)$  and  $c_{\{b, c\}}(f) = (2, 2)$ .  $\square$

Take a connected graph  $G$  of order  $n$ , and assume that  $V(G) = \{1, \dots, n\}$ . Let  $G[H^{(i)}]$  denote the graph obtained from  $G$  by replacing vertex  $i$  by a given graph  $H$  and joining every vertex of  $H$  to every neighbor of vertex  $i$  in  $G$ . Similarly,  $G[H_1^{(i)}, H_2^{(j)}]$  denotes the graph obtained from  $G$  by replacing vertex  $i$  by a graph  $H_1$  and vertex  $j$  by a graph  $H_2$  and joining every vertex of  $H_1$  (resp. vertex of  $H_2$ ) to every neighbor of vertex  $i$  (resp.  $j$ ) in  $G$  and, just if  $ij \in E(G)$ , also every vertex of  $H_1$  to every vertex of  $H_2$ . Finally,  $B$  denotes the bull graph shown in Figure 2.

**Theorem 2.** For any doubly-connected graph  $G$  with  $n \geq 4$ ,  $2 \leq \beta(G) + \beta(\overline{G}) \leq 2n - 6$ . Moreover,

- $\beta(G) + \beta(\overline{G}) = 2$  if and only if  $G = P_4$ .
- $\beta(G) + \beta(\overline{G}) = 2n - 6$  if and only if  $G \in \Omega_1 \cup \Omega_2 \cup \Omega_3$ , where
  - $\Omega_1 = \{P_4, C_5, B\}$
  - $\Omega_2 = \{P_4[K_{n-3}^{(1)}], P_4[\overline{K}_{n-3}^{(1)}], P_4[K_{n-3}^{(2)}], P_4[\overline{K}_{n-3}^{(2)}]\}$
  - $\Omega_3 = \{P_4[K_r^{(1)}, K_{n-r-2}^{(2)}] : 1 \leq r \leq n-3\} \cup \{P_4[\overline{K}_r^{(1)}, \overline{K}_{n-r-2}^{(3)}] : 1 \leq r \leq n-3\}$

*Proof.* In [3], it was proved that a connected graph  $G$  satisfies  $n-2 \leq \beta(G) \leq n-1$  if and only if, for some  $1 \leq h \leq n-1$ ,  $G \in \{K_n, K_{h, n-h}, K_h + \overline{K}_{n-h}, K_h + (K_1 \cup K_{n-h-1})\}$ . It is a routine exercise to check that the complement of any of these

graphs is not connected. Hence, every doubly-connected graph  $G$  of order  $n \geq 4$  satisfies  $1 \leq \beta(G) \leq n - 3$ , i.e.,  $2 \leq \beta(G) + \beta(\overline{G}) \leq 2n - 6$ . Moreover, according to Theorem 1, the lower bound 2 is attained only for  $G = P_4$ , since  $\overline{P_4} = P_4$ .

Let  $G$  be a doubly-connected graph of order  $n \geq 4$  verifying  $\beta(G) + \beta(\overline{G}) = 2n - 6$ , i.e., such that  $\beta(G) = \beta(\overline{G}) = n - 3$ . In [3], it was proved that the order of a graph  $G$  of diameter  $D$  and location number  $\beta$  is at least  $\beta + D$ . This means, that if  $\beta(G) = n - 3$ , then  $2 \leq D \leq 3$ , since  $\beta(K_n) = n - 1$ . In [11], the set of graphs with  $n$  vertices, diameter  $D$  and location number  $n - D$  were characterized for all feasible values of  $n$  and  $D$ . In particular, we have the set of graphs with  $n \geq 4$  vertices, diameter  $\text{diam}(G) = D = 3$  and location number  $n - 3$ , all of them being doubly-connected and verifying  $\text{diam}(\overline{G}) = 3$ . Among them, we are just interested in those graphs  $G$  for which  $\beta(\overline{G}) = n - 3$ . It is a routine exercise to check that as well as the path  $P_4$  and the bull graph  $B$ , the only doubly-connected graphs of diameter 3 satisfying  $\beta(G) = \beta(\overline{G}) = n - 3$  are those belonging to  $\Omega_3 \cup \Omega_3$ . Hence, according to Lemma 1, to finalize the proof it suffices to check that the only doubly-connected graph of order  $4 \leq n \leq 5$  having both itself and its complement diameter 2 is the cycle  $C_5$ .  $\square$

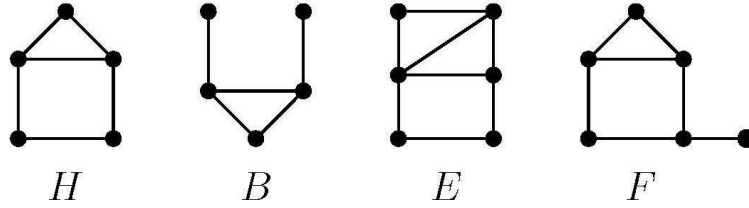


Figure 2: House graph  $H = \overline{P_5}$ , bull graph  $B = \overline{B}$ , graph  $E$  and graph  $F = \overline{E}$ ...

## 2.2. Metric-location-domination number

**Theorem 3.** For every nontrivial graph  $G$ ,  $3 \leq \eta(G) + \eta(\overline{G}) \leq 2n - 1$ . Moreover,

- $\eta(G) + \eta(\overline{G}) = 3$  if and only if  $\{G, \overline{G}\} = \{K_2, \overline{K_2}\}$ .
- $\eta(G) + \eta(\overline{G}) = 2n - 1$  if and only if  $\{G, \overline{G}\} = \{K_n, \overline{K_n}\}$ .

*Proof.* The only nontrivial graph  $G$  such that  $\eta(G) = 1$  is  $G = K_2$ , which means that for every graph  $G$ ,  $3 \leq \eta(G) + \eta(\overline{G})$ . Moreover, the equality  $\eta(G) + \eta(\overline{G}) = 3$  is only true when either  $G$  or  $\overline{G}$  is  $K_2$ , since  $\eta(\overline{K_2}) = 2$ . The rest of the proof is similar to that of Theorem 1.  $\square$

Given two positive integers  $r, s$ , let  $K_2(r, s)$  denote the so-called double star, obtained after joining the central vertices of the stars  $K_{1,r}$  and  $K_{1,s}$ . If  $2 \leq s \leq r - 1$ , let  $K_{1,r}^s$  represent the graph obtained by adding a new vertex adjacent to  $s$  leaves of the star  $K_{1,r}$ . Finally,  $\overline{K_2}(r, s)$ ,  $\overline{K_{1,r}}^s$  denote the complements of  $K_2(r, s)$ ,  $K_{1,r}^s$ , respectively, and graphs  $B, H, E$  and  $F$  are shown in Figure 2.

**Theorem 4.** For any doubly-connected graph  $G$  with  $n \geq 5$ ,  $4 \leq \eta(G) + \eta(\overline{G}) \leq 2n - 5$ . Moreover,

- $\eta(G) + \eta(\overline{G}) = 4$  if and only if  $G \in \{P_5, C_5, B, H, E, F\}$ .
- $\eta(G) + \eta(\overline{G}) = 2n - 5$  if and only if  $G \in \{K_2(r, s), \overline{K}_2(r, s), K_{1,r}^s, \overline{K}_{1,r}^s\}$ .

*Proof.* Every doubly-connected graph  $G$  of order at least 5 satisfies  $2 \leq \eta(G)$ , since the unique nontrivial graph such that  $\eta(G) = 1$  is  $G = P_2$ . In other words, for every nontrivial doubly-connected graph  $G$ ,  $4 \leq \eta(G) + \eta(\overline{G})$ . In [2], it was proved that there are exactly 51 connected graphs satisfying  $\eta(G) = 2$ , any of them having an order between 3 and 8. It is a routine exercise to check that the only doubly-connected graphs  $G$  with order at least 5 of this family whose complement verify also  $\eta(\overline{G}) = 2$  are exactly the graphs belonging to the set  $\{P_5, C_5, B, H, E, F\}$ .

In [10], it was proved that if  $G$  is a connected graph such that  $\eta(G) = n - 1$ , then  $G$  is either the complete graph  $K_n$  or the star  $K_{1,n-1}$ . Hence, every doubly-connected graph  $G$  of order  $n \geq 4$  satisfies  $\eta(G) \leq n - 2$ , since both  $\overline{K}_n$  and  $\overline{K}_{1,n-1}$  are not connected. Also in [10], all connected graphs  $G$  for which  $\eta(G) = n - 2$  were completely characterized. It is a routine exercise to check that the complement of any graph  $G$  verifying  $\eta(G) = n - 2$  is not connected unless  $G$  is either a double star  $K_2(r, s)$  or a graph  $K_{1,r}^s$ . As  $\eta(\overline{K}_2(r, s)) = \eta(\overline{K}_{1,r}^s) = n - 3$ , we conclude first, that every doubly-connected graph  $G$  of order  $n \geq 5$  satisfies  $\eta(G) + \eta(\overline{G}) \leq 2n - 5$  and second, that these four families are the only ones attaining this upper bound. □

### 2.3. Location-domination number

**Theorem 5.** For every nontrivial graph  $G$ ,  $3 \leq \lambda(G) + \lambda(\overline{G}) \leq 2n - 1$ . Moreover,

- $\lambda(G) + \lambda(\overline{G}) = 3$  if and only if  $\{G, \overline{G}\} = \{K_2, \overline{K}_2\}$ .
- $\lambda(G) + \lambda(\overline{G}) = 2n - 1$  if and only if  $\{G, \overline{G}\} = \{K_n, \overline{K}_n\}$ .

*Proof.* It is similar to that of Theorem 3. □

**Theorem 6.** For any doubly-connected graph  $G$  with  $n \geq 5$ ,  $4 \leq \lambda(G) + \lambda(\overline{G}) \leq 2n - 5$ . Moreover,

- $\lambda(G) + \lambda(\overline{G}) = 4$  if and only if  $G \in \{P_5, C_5, B, H\}$ .
- $\lambda(G) + \lambda(\overline{G}) = 2n - 5$  if and only if  $G \in \{K_2(r, s), \overline{K}_2(r, s), K_{1,r}^s, \overline{K}_{1,r}^s\}$ .

*Proof.* Every doubly-connected graph  $G$  of order at least 5 satisfies  $2 \leq \lambda(G)$ , since the unique nontrivial graph such that  $\lambda(G) = 1$  is  $G = P_2$ . In other words, for every nontrivial doubly-connected graph  $G$ ,  $4 \leq \lambda(G) + \lambda(\overline{G})$ . In [2], it was proved that there are exactly 16 connected graphs satisfying  $\lambda(G) = 2$ , any of them having an order between 3 and 5. It is a routine exercise to check that the only doubly-connected graphs  $G$  of this family whose complement verify also  $\lambda(\overline{G}) = 2$  are the 5-path  $P_5$ , the 5-cycle  $C_5$ , the bull graph  $B$  and the house graph  $H$  (see Figure 2). The rest of the proof is similar to that of Theorem 4 since for every graph  $G$ , if  $\lambda(G) = n - 1$ , then  $G$  is either the complete graph  $K_n$  or the star  $K_{1,n-1}$  [15] and,  $\lambda(G) = n - 2$  if and only if  $\eta(G) = n - 2$  [2]. □

Observe that the only doubly-connected graph of order at most 4 is  $P_4$ , and notice also that  $\bar{P}_4 = P_4$  and  $\eta(P_4) = \lambda(P_4)$ , which means that  $\eta(P_4) + \eta(\bar{P}_4) = \lambda(P_4) + \lambda(\bar{P}_4) = 4$ .

Finally, we present a further Nordhaus-Gaddum-type result for the parameter  $\lambda$ , which is a direct consequence of the fact that LD-sets in a graph  $G$  are very strongly related to LD-sets in its complement  $\bar{G}$ .

**Proposition 1.** *If  $S$  is an LD-set of a graph  $G$  then  $S$  is also an LD-set of  $\bar{G}$ , unless there exists a vertex  $w \in V \setminus S$  such that  $S \subseteq N_G(w)$ , in which case  $S \cup \{w\}$  is an LD-set of  $\bar{G}$ .*

*Proof.* Take  $u, v \in V \setminus S$ . Since  $S$  is an LD-set of  $G$ ,  $\emptyset \neq S \cap N_G(u) \neq S \cap N_G(v) \neq \emptyset$ . Hence,  $S \cap N_{\bar{G}}(u) = S \setminus S \cap N_G(u) \neq S \setminus S \cap N_G(v) = S \cap N_{\bar{G}}(v)$ . At this point we distinguish two cases: if there exists a vertex  $w \in V \setminus S$  such that  $S \subseteq N_G(w)$ , or equivalently, such that  $S \cap N_{\bar{G}}(w) = \emptyset$ , then it is unique as  $c_S(w) = (1 \dots 1)$ , and thus  $S \cup \{w\}$  is an LD-set. Otherwise, for every vertex  $w$ ,  $S \cap N_{\bar{G}}(w) \neq \emptyset$ , which means that  $S$  is also an LD-set of  $\bar{G}$ .  $\square$

**Theorem 7.** *For every graph  $G$ ,  $|\lambda(G) - \lambda(\bar{G})| \leq 1$ .*

*Proof.* According to Proposition 1, if  $S$  is a  $\lambda$ -code of  $G$ , then there exists an LD-set of  $\bar{G}$  of cardinality at most  $\lambda(G) + 1$ , which means that  $\lambda(\bar{G}) \leq \lambda(G) + 1$ . Similarly, it is derived that  $\lambda(G) \leq \lambda(\bar{G}) + 1$ , as  $G = \bar{\bar{G}}$ .  $\square$

**Corollary 1.** *Every graph  $G$  satisfies:  $2\lambda(G) - 1 \leq \lambda(G) + \lambda(\bar{G}) \leq 2\lambda(G) + 1$ .*

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